PERMUTATIONS & COMBINATIONS

Fundamental Principle of multiplication

A procedure with m first-stage outcomes and n second-stage outcomes can be executed in **mn ways**.

- **e.g.** 12 gates to hall, exiting from different gate
  - Total no. of ways : $12 \times 11 = 132$ Ways

Factorial Notation : n!

- Product of several consecutive integers
  - $n! = n.(n-1).(n-2).(n-3)....3.2.1$
  - $n! = n.(n-1)! = n.(n-1).(n-2)!$
  - $(2n)! = 2^n.\ n!\ [1.3.5.7....(2n-1)]$
  - $n.(n-1).(n-r+1) = n!/(n-r)!$

Permutations

- Arrangement of objects in a specific order, where the order of arrangement matters.

$$^{n}P_{r} = \frac{n!}{(n-r)!} = n(n-1)(n-2) \text{ .... } (n-r+1)$$

Total objects : **n** ; Objects taken : **r** $(0 \leq r \leq n)$

- All objects taken at once : Ways : $^{n}P_{n} = n!$

- **e.g.** 8 People in competition : Ways to give gold, silver and bronze medal = $^{8}P_{3} = 336$ ways

Permutation with REPETITION involved

- n different objects taken r at a time when **each object** may be repeated any no. of times is $\mathbf{n^r}$.

e.g. 7 letters and 5 letter-boxes, Total ways to put letters

Every letter has 5 boxes to choose : $\mathbf{5^7}$

Exhaustive approach : take 'n' as the object which is going to definitely exhaust while performing event. The letters in above case are going to be exhausted.

Permutation of ALIKE Objects

- n different objects all at once p objects are of 1st kind, q objects are of 2nd kind, then ways : **n!/(p!q!)**

e.g. Ways to arrange 'SWALLOW' = 7!/(2!2!) = 1260

Permutation under restriction

- n different objects, taken r at a time, Particular object is to be always included in each arrangement = Total ways = $\mathbf{r.\ {}^{n-1}P_{r-1}}$
- n different objects, taken r at a time, Particular object is to be never included in arrangement = Total ways = $\mathbf{{}^{n-1}P_r}$

String method

- n different objects taken all at a time when m objects always occur together : **m! × (n – m + 1)!.**

e.g. number of ways to arrange 'VIBGYOR' when **red, yellow and orange** comes together : 3! (7-3+1)! = 720

Gap Method

- number of permutations when no two given objects occur together : **$m! \times {}^{m+1}C_n \times n!$**

1. e.g. number of ways to arrange 'VIBGYOR' when red, yellow and orange never comes together

Possible arrangement : | V | I | B | G |

Arrangement (VIBG) x Gap Selectionx Arrangement (RYO)

$4! \times {}^5C_3 \times 3! = 1440$ ways

Combinations

- Selection of objects, where the order of arrangement doesn't matter.

$$C(n, r) \text{ or } {}^nC_r = \frac{n!}{r!(n-r)!} \quad \text{where, } (0 \le r \le n)$$

- Number of ways to select **r objects = (n-r) objects**
- Combination notation also represents the binomial coefficient.
- Clue words: group, committee, sample, selection, subset.

Properties of Combinations

(a) ${}^nC_r = {}^nC_{n-r}$

(b) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(c) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(d) If n is even, then the greatest value of ${}^nC_r$ is ${}^nC_{n/2}$

(e) If n is odd, then the greatest value of ${}^nC_r$ is ${}^nC_{\frac{n+1}{2}}$

(f) ${}^nC_0 + {}^nC_1 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n$

(g) ${}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots\ldots\ldots\ldots + {}^{2n-1}C_n = {}^{2n}C_{n+1}$

Combination under restriction

- Choosing r objects out of n different objects if p objects must be excluded : ${}^{n-p}C_r$
- Choosing r objects out of n different objects if p objects must be included : ${}^{n-p}C_{r-p}$
- Combinations of n different objects taken one or more at a time = $2^n - 1$.
- **e.g.** Out of 16 Cricket players, 11 are to be selected but keeper and captain are already included.
  - ${}^{16-2}C_{11-2} = {}^{14}C_9$
- **e.g.** Out of 16 Cricket players, 11 are to be selected but one player is injured.
  - ${}^{16-1}C_{11} = {}^{15}C_{11}$

Combination of ALIKE Objects

- If out of (p+q+r+s) objects, p are alike of one kind, q are alike of a second kind, r are alike of the third kind and s are different, then the total number of combinations is $\mathbf{(p + 1)(q + 1)(r + 1)2^s - 1}$

Division into Groups

- (m + n) different objects can be divided into two unequal groups containing m and n objects **without order** respectively is **(m+n)!/m!n!**
  - if **m=n**, (2n)!/n!n!2! as it is possible to interchange the two groups **(without order)**
  - when order is considered, multiply by factorial of no. of groups

- e.g. 12 balls to be distributed between two boys receiving 5 balls to one, 7 balls to other.
  - Ways to select distribute : 12!/5!7! = 792
  - Since order is important : 792 x 2! = 1584

Geometrical Problems in P&C

Lines

• n points in a plane, no 3 points are collinear, no. of lines formed	${}^nC_2$
• n points in a plane, m points are collinear, no. of lines formed	${}^nC_2 - {}^mC_2 + 1$

Geometrical Problems in P&C	
Triangles	
• n (non-collinear) points in a plane no. of Triagnles formed	${}^nC_3$
• n points in a plane, m points are collinear, no. of Triangles formed	${}^nC_3 - {}^mC_3$
Diagonals	
• no. of diagonals in a n side polygon	${}^nC_2 - n$
Rectangles and Squares	
• no. of rectangles in a n x n sized square	$n^2(n+1)^2/4$
• no. of squares in a n x n sized square	$n(n+1)(2n+1)/6$
• no. of rectangles in a m x n sized rectangle	$(mn/4)(m+1)(n+1)$
• no. of squares in a m x n sized rectangle	$\sum_{k=0}^{n}(m-k+1)(n-k+1)$

De-Arrangement Problems (Letter problem)

n objects arranged in m places, no. of ways to de-arrange objects so that none occupies original place

$$D_n = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!} = n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\ldots\ldots+(-1)^n\frac{1}{n!}\right)$$

Exponent of a prime number	$e_p(n) = \sum_{i=1}^{\infty}\left[\frac{n}{p^i}\right]e_p$

- **e.g.** Exponent of 2 in 100!

$$\left[\frac{100}{2}\right]=50;\left[\frac{100}{2^2}\right]=25;\left[\frac{100}{2^3}\right]=12;\left[\frac{100}{2^4}\right]=6;\left[\frac{100}{2^5}\right]=3;\left[\frac{100}{2^6}\right]=1$$

next is 0, thus exponent is : 50+25+12+6+3+1 = 97

Sum of Numbers

- For $a_1, a_2, a_3 \dots an$, the sum of the digits in the units place of all numbers formed (if numbers are not repeated) is
  - $(a_1 + a_2 + a_3 + \dots + a_n)(n-1)!$
    - i.e. (sum of the digits) $(n-1)!$
- Sum of the total numbers which can be formed with given different digits $a_1, a_2, a_3 \dots\dots a_n$ is
  - $(a_1 + a_2 + a_3 + \dots + a_n)(n-1)!$ (111. ...... n times)

Circular Permutation (objects all together)

- The number of circular permutations of n different objects altogether
- $nPn/n = (n-1)!$, when clockwise and anticlockwise order are treated as different,
- $nPn/2n = 1/2(n-1)!$, when the above two orders are treated as same.